3901. (a) Setting $x = 0$ gives $t = 3$, which we verify also gives $y = 0$. The velocities are

$$\begin{aligned}\dot{x} &= 3(t-3)^2,\\ \dot{y} &= 3(t-2)^2 - 1.\end{aligned}$$

At $t = 3$ these are $\dot{x} = 0$ and $\dot{y} = 2$. So, the particle is moving in the positive y direction as it passes through the origin.

(b) Setting $y = 0$, we take out a factor of $(t-2)$:

$$\begin{aligned}&(t-2)^3 - (t-2) = 0\\ \Longrightarrow\ &(t-2)\big((t-2)^2 - 1\big) = 0\\ \Longrightarrow\ &t = 1, 2, 3.\end{aligned}$$

At $t = 1$, the particle is at $(-8, 0)$ and it has velocities $\dot{x} = 12$ and $\dot{y} = 2$. So, the angle at which it crosses is $\arctan \frac{2}{12} = \arctan \frac{1}{6}$.

3902. Assume, for a contradiction, that $\sqrt{2} - \sqrt{3} = p/q$, where p and q are integers. Squaring both sides,

$$\begin{aligned}&\left(\sqrt{2} - \sqrt{3}\right)^2 = \frac{p^2}{q^2}\\ \Longrightarrow\ &2 - 2\sqrt{6} + 3 = \frac{p^2}{q^2}\\ \Longrightarrow\ &\sqrt{6} = \frac{5q^2 - p^2}{2q^2}.\end{aligned}$$

The numerator and denominator of the RHS are integers, which implies that $\sqrt{6}$ is rational. But 6 is not a perfect square. This is a contradiction. Hence, $\sqrt{2} - \sqrt{3}$ is irrational. □

3903. The set $A \cap B$ is $\{n \in \mathbb{Z} : a < n < b\}$, which is the integers between a and b, not inclusive.

(a) The number of elements in $\{n \in \mathbb{Z} : a < n < b\}$ is $b - a - 1$. So, $b - a = 21$.

(b) The lowest possible value of a is 1, which gives $b = 22$. The highest possible value of b is 100, which gives $a = 79$. Hence, the set of possible values for $a + b$ is $\{k \in \mathbb{Z} : 23 \le k \le 179\}$.

3904. The denominator, when written multiplicatively, is $\left(4 + qx^2\right)^{-3}$. Taking out $4^{-3} = \frac{1}{64}$ leaves

$$\begin{aligned}&\left(1 + \tfrac{qx^2}{4}\right)^{-3}\\ \equiv\ &1 + (-3)\tfrac{qx^2}{4} + \tfrac{(-3)(-4)}{2!}\left(\tfrac{qx^2}{4}\right)^2 + ...\\ \equiv\ &1 - \tfrac{3q}{4}x^2 + \tfrac{3q^2}{8}x^4 + ...\end{aligned}$$

So, the LHS of the original identity is

$$\begin{aligned}&\tfrac{p}{4^3}\left(1 - \tfrac{3q}{4}x^2 + \tfrac{3q^2}{8}x^4 + ...\right)\\ \equiv\ &\tfrac{p}{4^3} - \tfrac{3pq}{256}x^2 + \tfrac{3pq^2}{512}x^4 + ...\end{aligned}$$

Equating coefficients in the original identity,

$$\begin{aligned}x^0 &: \tfrac{p}{4^3} = 1\\ x^2 &: -\tfrac{3pq}{256} = \tfrac{3}{4}\\ x^4 &: \tfrac{3pq^2}{512} = r.\end{aligned}$$

The first equation gives $p = 64$. Then the second gives $q = -1$. Then the third gives $r = \frac{3}{8}$.

3905. (a) Multiplying out,

$$\begin{aligned}S &= n^3 + (n+1)^3 + (n+2)^3\\ &\equiv 3n^3 + 9n^2 + 15n + 9.\end{aligned}$$

Using a polynomial solver, this has a root at $n = -1$, so a factor of $(n+1)$. Taking this and a factor of 3 out, $S = 3(n+1)(n^2 + 2n + 3)$.

(b) Consider including or excluding, in a product, each of the factors

$$\left\{3, (n+1), (n^2 + 2n + 3)\right\}.$$

This gives 2^3 possible factors, from 1 (all three excluded) to S (all three included). We must show that these eight are distinct.

We are told that the cubes are greater than eight, so that $2 < n$. This gives

$$3 < n + 1 < n^2 + 2n + 3.$$

Also, we need to check that the product $3n+3$ of the two smaller factors is distinct from the larger one. Since $3 < n+1$, we know that

$$\begin{aligned}3n + 3 &< n(n+1) + 3\\ &\equiv n^2 + 2n + 3.\end{aligned}$$

Hence, the sum of three distinct cubes over 8 has at least eight distinct factors. □

3906. A queen threatens a different number of squares, depending on her location. She always threatens $7 + 7 = 14$ squares in her row and column; the number threatened diagonally varies. According to distance from the edge, the number of squares, and the number threatened, are as follows:

Distance	Number	Threatened
0	28	$14 + 7 = 21$
1	20	$14 + 9 = 23$
2	12	$14 + 11 = 25$
3	4	$14 + 13 = 27$

There are 64 squares. So, the probability that two randomly placed queens threaten each other is

$$p = \tfrac{28}{64} \cdot \tfrac{21}{63} + \tfrac{20}{64} \cdot \tfrac{23}{63} + \tfrac{12}{64} \cdot \tfrac{25}{63} + \tfrac{4}{64} \cdot \tfrac{27}{63}.$$

This gives $p = \frac{13}{36}$.

3907. The derivative is

$$\frac{dy}{dx} = 4x^3 - 1.$$

At $(1, 0)$, the gradient is $m = 3$. So, the tangent is $y = 3x - 3$. Solving for intersections,

$$\begin{aligned} &x^4 - x = 3x - 3 \\ \Longrightarrow\ &x^4 - 4x + 3 = 0. \end{aligned}$$

We know that this has a double root at $x = 1$, so we take out a factor of $(x-1)^2$:

$$\begin{aligned} &x^4 - 4x + 3 \\ \equiv\ &(x-1)^2(x^2 + 2x + 3). \end{aligned}$$

The quadratic factor has $\Delta = -8 < 0$, so there are no other intersections, as required.

3908. Consider any two distinct rational numbers a and b, with $a < b$. The number $b - a$ is positive and rational, so can be written as p/q, where p and q are integers. Consider

$$\delta = \frac{\sqrt{2}}{2} \times \frac{p}{q}.$$

This is irrational, and, since $\sqrt{2}/2 < 1$, is smaller than $b - a$. Hence, $a + \delta$ is an irrational number between a and b. QED.

3909. (a) Differentiating implicitly,

$$\frac{1 + \frac{dy}{dx}}{2\sqrt{x+y}} + \frac{1 - \frac{dy}{dx}}{2\sqrt{x-y}} = 0.$$

Multiplying by $2\sqrt{x-y}\sqrt{x+y}$, this is

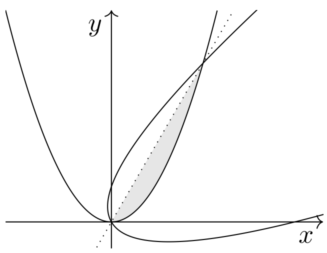
$$\left(1 + \tfrac{dy}{dx}\right)\sqrt{x-y} + \left(1 - \tfrac{dy}{dx}\right)\sqrt{x+y} = 0.$$

(b) Setting gradient $\frac{dy}{dx} = 1$ gives

$$2\sqrt{x-y} = 0.$$

So, $x = y$. Substituting this into the equation of the curve, $\sqrt{2x} = 1$, so $x = 1/2$. Hence, the coordinates are $(1/2, 1/2)$.

3910. (a) The curves are as shown:



Since C_2 has been rotated 60° clockwise, the line of symmetry between C_1 and C_2 has been rotated by 30° clockwise from the y axis. Hence, its equation is $y = (\tan 60°)x$, which is $y = \sqrt{3}x$.

(b) Solving for the point of intersection, $x^2 = \sqrt{3}x$, which gives O and the point $x = \sqrt{3}$. So, the area (shaded) enclosed by the line and C_1 is

$$\begin{aligned} &\int_0^{\sqrt{3}} \sqrt{3}x - x^2\, dx \\ =\ &\left[\tfrac{\sqrt{3}}{2}x^2 - \tfrac{1}{3}x^3\right]_0^{\sqrt{3}} \\ =\ &\tfrac{\sqrt{3}}{2}. \end{aligned}$$

The area enclosed by C_1 and C_2 is double this, which is $\sqrt{3}$, as required.

3911. The forces in the interaction are a Newton III pair, so, as vectors, they can be written F and $-F$. The accelerations are

$$\begin{aligned} a_1 &= \frac{F}{m_1}, \\ a_2 &= \frac{-F}{m_2}. \end{aligned}$$

So, the changes in velocity are given by

$$\begin{aligned} v_1 - u_1 &= \frac{Ft}{m_1}, \\ v_2 - u_2 &= \frac{-Ft}{m_2}. \end{aligned}$$

Multiplying by m_1 and m_2 gives

$$\begin{aligned} m_1v_1 - m_1u_1 &= Ft, \\ m_2v_2 - m_2u_2 &= -Ft. \end{aligned}$$

Adding these equations,

$$\begin{aligned} &m_1v_1 - m_1u_1 + m_2v_2 - m_2u_2 = 0 \\ \Longrightarrow\ &m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2. \end{aligned}$$

Total momentum before interaction (LHS) is equal to total momentum after interaction (RHS), which is conservation of momentum. □

3912. By the product rule,

$$\frac{dy}{dx} = \mathrm{f}'(x)e^{x^2} + 2x\,\mathrm{f}(x)e^{x^2}.$$

Substituting into the DE,

$$\begin{aligned} &\mathrm{f}'(x)e^{x^2} + 2x\,\mathrm{f}(x)e^{x^2} = 2x\,\mathrm{f}(x)e^{x^2} \\ \Longrightarrow\ &\mathrm{f}'(x)e^{x^2} = 0. \end{aligned}$$

Since $e^{x^2} \neq 0$, we know $\mathrm{f}'(x) = 0$. Integrating, this implies $\mathrm{f}(x) = c$, as required.

3913. (a) Dividing top and bottom by x^2, we have

$$\lim_{x\to\infty} \frac{600 - 1470\frac{1}{x} + 441\frac{1}{x^2}}{50 - 135\frac{1}{x} + 63\frac{1}{x^2}}.$$

Taking the limit, the inlaid fractions tend to zero, giving $\frac{600}{50} = 12$.

(b) Factorising top and bottom, we have

$$\begin{aligned} &\lim_{x\to\frac{21}{10}} \frac{(10x-21)(60x-21)}{(10x-21)(5x-3)} \\ = &\lim_{x\to\frac{21}{10}} \frac{60x-21}{5x-3} \\ = &\; 14. \end{aligned}$$

3914. The curve is increasing, so $4x^3 - 2x > 0$. This has boundary equation $x(2x^2-1) = 0$, so roots $x = 0, \pm 1/\sqrt{2}$. As a positive cubic, the inequality has solution $x \in (-1/\sqrt{2}, 0) \cup (1/\sqrt{2}, \infty)$.

The curve is concave, so $12x^2 - 2 < 0$. This has boundary equation $6x^2 - 1 = 0$, so roots $x = \pm 1/\sqrt{6}$. As a positive quadratic, the inequality has solution $x \in (-1/\sqrt{6}, 1/\sqrt{6})$.

To be in both solution sets, $x \in (-1/\sqrt{6}, 0)$.

3915. Since x is small, $\sin x \approx x$. This gives

$$\begin{aligned} &\frac{1}{1-\sin x + 2\sin^2 x} \\ \approx &\frac{1}{1-x+2x^2} \\ = &\big(1 + (-x+2x^2)\big)^{-1}. \end{aligned}$$

Using the generalised binomial expansion, this is

$$\begin{aligned} &1 + (-1)\big(-x+2x^2\big) + \tfrac{(-1)(-2)}{2!}\big(-x+2x^2\big)^2 + ... \\ \equiv &\; 1 + x - 2x^2 + x^2 + ... \\ \equiv &\; 1 + x - x^2 + ..., \text{ as required.} \end{aligned}$$

3916. (a) Differentiating with respect to y,

$$\begin{aligned} & xy - x - y^2 = 0 \\ \Longrightarrow\; & \tfrac{dx}{dy}y + x - \tfrac{dx}{dy} - 2y = 0. \end{aligned}$$

Setting $\frac{dx}{dy} = 0$ gives $2y = x$. Subbing this back into the equation of the curve,

$$\begin{aligned} & 2y^2 - 2y - y^2 = 0 \\ \Longrightarrow\; & y = 0, 2. \end{aligned}$$

Hence, the points at which the gradient $\frac{dy}{dx}$ is undefined are $(0,0)$ and $(4,2)$.

(b) Rearranging to make x the subject,

$$x = \frac{y^2}{y-1}.$$

The denominator is zero at $y = 1$, so this is the equation of the horizontal asymptote. The coordinates of P are $(2,1)$.

(c) The line $y - 1 = m(x-2)$ passes through P with gradient m. Working visually, this line will intersect the curve if it is shallower than the oblique asymptote and steeper than the horizontal. So, $m \in (0,1)$.

3917. Let $z = \sin x$ and $y = \tan x$. Then the rate of change of $\tan x$ with respect to $\sin x$ is

$$\begin{aligned} \frac{dz}{dy} &\equiv \frac{dz}{dx} \div \frac{dy}{dx} \\ &= \cos x \div \sec^2 x \\ &\equiv \cos^3 x. \end{aligned}$$

There are values x for which $\cos^3 x = 0$. However, since $\tan x \equiv \frac{\sin x}{\cos x}$, none of those values are in the domain of $\tan x$. Hence, $\tan x$ is never stationary with respect to $\sin x$. □

3918. Using a polynomial solver, the roots are

$$x = -\tfrac{1}{2}, -\tfrac{1}{3}, 1, 2.$$

Two of these are positive. Hence, when choosing (without replacement) two roots, the probability p that exactly one of the roots is positive is

$$p = 2 \cdot \tfrac{2}{4} \cdot \tfrac{2}{3} = \tfrac{2}{3}.$$

3919. The discriminant is $\Delta = -72 < 0$, so the equation has no real roots. Subbing $z = 4 \pm 2i$ into the LHS,

$$\begin{aligned} & (4 \pm 2i)^2 - 8(4 \pm 2i) + 20 \\ = &\; 16 \pm 16i + 4i^2 - 32 \mp 16i + 20 \\ = &\; 16 \pm 16i - 4 - 32 \mp 16i + 20 \\ = &\; 0, \text{ as required.} \end{aligned}$$

— Nota Bene —

The set of complex numbers $\mathbb{C}$, which includes but is broader than $\mathbb{R}$, is a huge mathematical idea, lying immediately beyond the scope of this book. Much mathematics beyond this point (including, for example, all of quantum physics) is based on the set of complex numbers $\mathbb{C}$. It is a 2D number plane, broadening the 1D number line.

3920. (a) For $x \to -\infty$, $e^x \to 0$, so $y \to -1$. For $x \to \infty$, we first divide top and bottom by e^x, giving

$$y = \frac{1 - e^{-x}}{1 + e^{-x}}.$$

As $x \to \infty$, $e^{-x} \to 0$, so $y \to 1$.

(b) Differentiating by the quotient rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{e^x(e^x+1) - e^x(e^x-1)}{(e^x+1)^2} \\ &\equiv \frac{2e^x}{(e^x+1)^2}. \end{aligned}$$

Since $e^x > 0$, the numerator is never zero, so there are no stationary points.

(c) Suppose that (a, b) is on the curve, so that

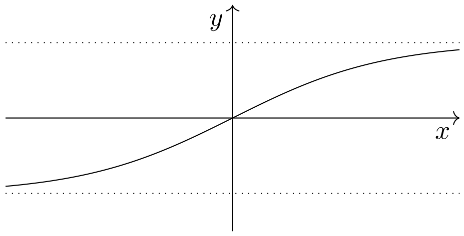
$$b = \frac{e^a - 1}{e^a + 1}.$$

Instead substituting $x = -a$ into the RHS,

$$\begin{aligned} &\frac{e^{-a} - 1}{e^{-a} + 1} \\ \equiv\ &\frac{1 - e^a}{1 + e^a} \\ \equiv\ &-\frac{e^a - 1}{e^a + 1} \\ \equiv\ &-b. \end{aligned}$$

Hence, $(-a, -b)$ is also a point on the curve. This means that the curve has odd symmetry, i.e. rotational symmetry around the origin.

(d) Using parts (a), (b) and (c), the graph is



3921. Putting the fractions over a common denominator,

$$\begin{aligned} &\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x} \\ \equiv\ &\frac{2}{1 - \sin^2 x} \\ \equiv\ &\frac{2}{\cos^2 x} \\ \equiv\ &2 \sec^2 x. \end{aligned}$$

So, the integral is

$$\begin{aligned} &\int_0^{\frac{1}{6}\pi} 2 \sec^2 x \, dx \\ =\ &\Big[2 \tan x\Big]_0^{\frac{1}{6}\pi} \\ =\ &\tfrac{2\sqrt{3}}{3}, \text{ as required.} \end{aligned}$$

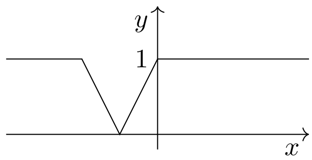
3922. Multiplying by $\ln x$, this is a cubic in $\ln x$:

$$3(\ln x)^3 - 13(\ln x)^2 + 16 \ln x - 4 = 0.$$

Using a polynomial solver, $\ln x = \frac{1}{3}, 2$, which gives $x = \sqrt[3]{e}, e^2$.

3923. (a) This is true. The outputs of f are elements of $[0, 1]$, and outputs of f^2 are outputs of f.

(b) This is false. Consider f as in the following graph of $y = f(x)$:



This has range $[0, 1]$ over $\mathbb{R}$. However, since inputs $x \in [0, 1]$ all produce the output 1, the function f^2 has range $\{1\}$.

3924. (a) Proceeding algebraically,

$$\begin{aligned} &\frac{x}{x + y} + \frac{x + y}{y} = k \\ \Longrightarrow\ &xy + (x + y)^2 = ky(x + y) \\ \Longrightarrow\ &x^2 + (3 - k)xy + (1 - k)y^2 = 0. \end{aligned}$$

(b) The above may be treated as a quadratic in x. Using the quadratic formula,

$$\begin{aligned} x &= \frac{-(3 - k)y \pm \sqrt{(3 - k)^2 y^2 - 4(1 - k)y^2}}{2} \\ &\equiv \frac{k - 3 \pm \sqrt{k^2 - 2k + 5}}{2} y. \end{aligned}$$

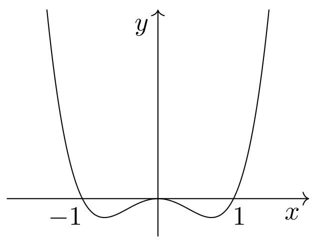
Since $k^2 - 2k + 5$ has $\Delta = -16 < 0$, it is always positive. Hence, the locus is of two equations $x = ay$ and $x = by$, where $a \neq b$. These are distinct lines through the origin, as required.

3925. (a) The two-way table is as follows.

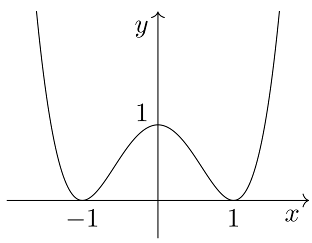
B gives \ A gives	J	J	Q
Q	✓	✓	✓ (shaded)
Q	✓	✓	✓ (shaded)
K	(shaded)	(shaded)	✓ (shaded)

(b) In the diagram above, the restricted possibility space is shaded, and the successful outcomes are ticked. This gives the probability as $\frac{3}{5}$.

3926. (a) The curve is $y = x^4 - x^2 \equiv x^2(x + 1)(x - 1)$. This has a double root at the origin, and single roots at $x = \pm 1$. The graph is



(b) The curve is $y = x^4 - 2x^2 + 1 \equiv (x+1)^2(x-1)^2$. This has double roots at $x = \pm 1$:



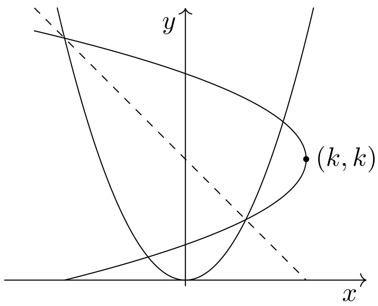
3927. Squaring both equations and also reciprocating the second,

$$x^2 = \operatorname{cosec}^2 2t,$$
$$y^{-2} = \cot^2 2t.$$

We substitute these into the third Pythagorean trig identity $1 + \cot^2 2t \equiv \operatorname{cosec}^2 2t$, giving

$$1 + y^{-2} = x^2$$
$$\implies y^2(x^2 - 1) = 1.$$

3928. The scenario is as follows:



Reflection in $x + y = 0$ produces $x = -y^2$. The vertex is then translated by vector $k\mathbf{i} + k\mathbf{j}$, which gives $x - k = -(y - k)^2$.

3929. (a) Coordinates of points on a unit semicircle are $(\cos\theta, \sin\theta)$. So, $y = \sin(\arccos x)$ converts from the x coordinate to the y coordinate. Hence, it is the equation of the unit semicircle, which is $y = \sqrt{1 - x^2}$.

(b) The third angle is $\theta = \pi - \arccos\frac{11}{14} - \arccos\frac{13}{14}$. Using the identity $\sin(\pi - x) \equiv \sin x$,

$$\sin\theta = \sin(\pi - \arccos\tfrac{11}{14} - \arccos\tfrac{13}{14})$$
$$= \sin(\arccos\tfrac{11}{14} + \arccos\tfrac{13}{14}).$$

Using a compound-angle formula, this is

$$\sin(\arccos\tfrac{11}{14})\cos(\arccos\tfrac{13}{14}) + \cos(\arccos\tfrac{11}{14})\sin(\arccos\tfrac{13}{14})$$
$$= \sqrt{1 - \tfrac{11}{14}^2} \cdot \tfrac{13}{14} + \tfrac{11}{14}\sqrt{1 - \tfrac{13}{14}^2}$$
$$= \tfrac{5\sqrt{3}}{14} \cdot \tfrac{13}{14} + \tfrac{11}{14} \cdot \tfrac{3\sqrt{3}}{14}$$
$$= \tfrac{\sqrt{3}}{2}, \text{ as required.}$$

3930. The sum of the integers from 1 to $3k$ is $\frac{3}{2}k(3k+1)$. The sum of those from 1 to $3k$ which are divisible by 3 is three times the sum of those from 1 to k. This is $\frac{3}{2}k(k+1)$. So, the required sum is

$$\tfrac{3}{2}k(3k+1) - \tfrac{3}{2}k(k+1)$$
$$\equiv \tfrac{3}{2}k\big(3k + 1 - (k+1)\big)$$
$$\equiv 3k^2, \text{ as required.}$$

3931. (a) $X \sim B(n, 1/n)$.

(b) The expected number of switches on is

$$\mathbb{E}(X) = np = n/n = 1.$$

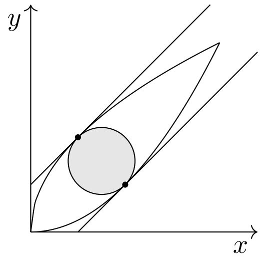
So, we need to calculate $\mathbb{P}(X = 1 \mid X \leq 1)$. To do this, we need

$$\mathbb{P}(X = 0) = \left(\tfrac{n-1}{n}\right)^n$$
$$\mathbb{P}(X = 1) = {}^n\mathrm{C}_1 \tfrac{1}{n}\left(\tfrac{n-1}{n}\right)^{n-1} \equiv \left(\tfrac{n-1}{n}\right)^{n-1}.$$

So, the required probability is

$$\mathbb{P}(X = 1 \mid X \leq 1) = \frac{\left(\frac{n-1}{n}\right)^{n-1}}{\left(\frac{n-1}{n}\right)^{n-1} + \left(\frac{n-1}{n}\right)^n}$$
$$\equiv \frac{1}{1 + \frac{n-1}{n}}$$
$$\equiv \frac{n}{2n-1}.$$

3932. The problem is symmetrical in $y = x$, so the centre lies on that line. Consider the boundary case, in which the circle is as large as possible. The points of tangency must be as far from $y = x$ as possible, so the circle must touch the curves at points with gradient 1. On $y = x^2$, this is at $2x = 1$. So, the points of tangency are at $(1/2, 1/4)$ and $(1/4, 1/2)$.



This gives the diameter of the circle as $\sqrt{2}/4$, and the radius as $\sqrt{2}/8$. Hence, the area of the circle must satisfy

$$A \leq \pi\left(\frac{\sqrt{2}}{8}\right)^2 = \frac{\pi}{32}, \text{ as required.}$$

3933. Substituting the latter into the former,

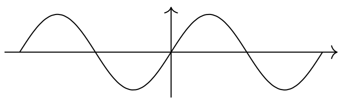
$$6 - 4q + \sqrt{4 + (6 - 4q)q} + q = 0$$
$$\implies \sqrt{4 + 6q - 4q^2} = 3q - 6$$
$$\implies 4 + 6q - 4q^2 = (3q - 6)^2$$
$$\implies 13q^2 - 42q + 32 = 0$$
$$\implies q = \tfrac{16}{13}, 2.$$

In squaring, we may have introduced new solution points. So, we check. Substituting $q = \frac{16}{13}$ into the second equation gives $p = \frac{14}{13}$. In the LHS of the first equation, this gives

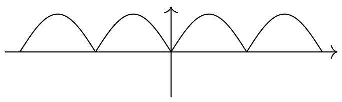
$$\tfrac{14}{13} + \sqrt{4 + \tfrac{16}{13} \cdot \tfrac{14}{13}} + \tfrac{16}{13} = \tfrac{60}{13} \neq 0.$$

So, $\left(\frac{14}{13}, \frac{16}{13}\right)$ does not satisfy the equations. Instead substituting $q = 2$, we get $p = -2$. These values satisfy both equations. The solution is $(-2, 2)$.

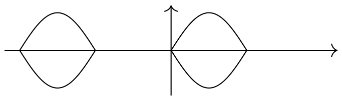
3934. (a) $y = \sin x$ is the usual sine graph:



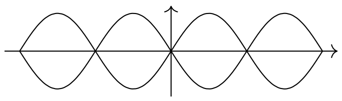
(b) In $y = |\sin x|$, all negative values of $\sin x$ in the above graph are rendered positive:



(c) In $|y| = \sin x$, where there are negative values of $\sin x$, the graph has no points, because $|y|$ must be positive. Where $\sin x \geq 0$, the y value can now be positive or negative.



(d) Algebraically, $|y| = |\sin x| \iff y = \pm \sin x$:



3935. (a) Using the quotient rule,

$$\begin{aligned} \frac{d}{dx}\left(\frac{\mathrm{f}_1(x)}{\mathrm{f}_2(x)}\right) &= \frac{\mathrm{f}_1'(x)\,\mathrm{f}_2(x) - \mathrm{f}_1(x)\,\mathrm{f}_2'(x)}{(\mathrm{f}_2(x))^2} \\ &= \frac{\mathrm{f}_1(x)\,\mathrm{f}_2(x) - \mathrm{f}_1(x)\,\mathrm{f}_2(x)}{(\mathrm{f}_2(x))^2} \\ &\equiv 0. \end{aligned}$$

(b) Integrating the above,

$$\begin{aligned} &\frac{d}{dx}\left(\frac{\mathrm{f}_1(x)}{\mathrm{f}_2(x)}\right) = 0 \\ \Longrightarrow\ &\frac{\mathrm{f}_1(x)}{\mathrm{f}_2(x)} = k \\ \Longrightarrow\ &\mathrm{f}_1(x) = k\,\mathrm{f}_2(x). \end{aligned}$$

So, any functions satisfying $\mathrm{f}'(x) = \mathrm{f}(x)$ must be proportional, as required.

3936. (a) Since $0 < a < b$, the range is $[1/b, 1/a]$.

(b) Writing the numerator as $\mathrm{f}(x) + 1 - 1$,

$$x \longmapsto 1 - \frac{1}{\mathrm{f}(x)+1}.$$

The denominator has range $[a+1, b+1]$. So, the fraction (without the minus sign) has range

$$\left[\tfrac{1}{b+1}, \tfrac{1}{a+1}\right].$$

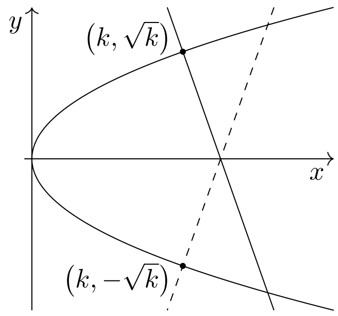
Subtracting this from 1 reverses the order of the boundaries, giving the range as

$$\left[1 - \tfrac{1}{a+1}, 1 - \tfrac{1}{b+1}\right].$$

This can be simplified back into the algebraic form of the original function, as

$$\left[\tfrac{a}{a+1}, \tfrac{b}{b+1}\right].$$

3937. The scenario is



There are two normals at $x = k$, as shown. By symmetry, they cross the x axis at the same point, so we can consider only the upper. Differentiating, $\frac{dx}{dy} = 2y$. At $\left(k, \sqrt{k}\right)$, this is $\frac{dx}{dy} = 2\sqrt{k}$. So, the equation of the normal is

$$y - \sqrt{k} = -2\sqrt{k}(x - k).$$

Substituting $y = 0$,

$$\begin{aligned} &-\sqrt{k} = -2\sqrt{k}(x-k) \\ \Longrightarrow\ &-1 = -2x + 2k \\ \Longrightarrow\ &x = k + \tfrac{1}{2}, \text{ as required.} \end{aligned}$$

3938. (a) The acceleration is $5g$ ms^{-2} downwards.

(b) Maximum range and minimum speed occur at $\theta = 45°$. Vertically and horizontally,

$$\begin{aligned} 0 &= u \sin 45° t - \tfrac{5}{2} g t^2, \\ 1 &= u \cos 45° t. \end{aligned}$$

Hence, $0 = 1 - \frac{5}{2}gt^2$, so $t = \sqrt{2}/7$. This gives

$$u_{\text{min}} = \frac{1}{\frac{\sqrt{2}}{7} \times \cos 45°} = 7.$$

The minimum speed is 7 ms^{-1}.

(c) Carrying out an equivalent calculation, with acceleration g ms^{-2}, time of flight is $t = \frac{\sqrt{10}}{7}$, so the minimum speed is 3.13 ms^{-1} (3sf).

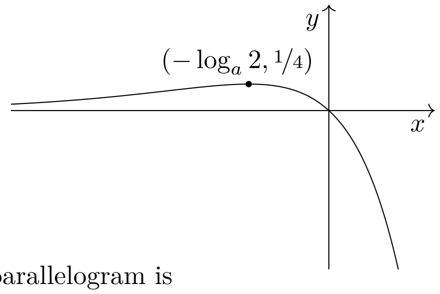
In an accelerating rocket, an object must be thrown faster, even relative to the moving cockpit, to attain the same horizontal range as on Earth.

3939. For x intercepts, $a^x - a^{2x} = 0$. This factorises as $a^x(1 - a^x) = 0$. Since $a^x \neq 0$, $a^x = 1$. So, the only axis intercept is the origin.

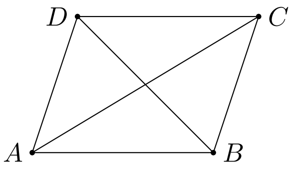
For SPs, using $\frac{d}{dx}a^x = \ln a \cdot a^x$,

$$\begin{aligned} &\ln a \cdot a^x - 2 \ln a \cdot a^2 x = 0 \\ \Longrightarrow\ &\ln a \cdot a^x(1 - 2a^x) = 0. \end{aligned}$$

Since $a > 1$, $\ln a \cdot a^x \neq 0$, so $a^x = \frac{1}{2}$. This gives a stationary point at $x = \log_a 1/2 \equiv -\log_a 2$. The second derivative is negative here, so $(-\log_a 2, 1/4)$ is a local maximum. As $x \to -\infty$, $y \to 1^+$. As $x \to \infty$, $y \to -\infty$. Putting all of this together, the graph is as follows:



3940. The parallelogram is



The cosine rule on $\triangle ABC$ is

$$|AC|^2 = |AB|^2 + |BC|^2 - 2|AB||BC|\cos\angle ABC.$$

And on $\triangle ABD$ is

$$|BD|^2 = |AB|^2 + |AD|^2 - 2|AB||AD|\cos\angle DAB.$$

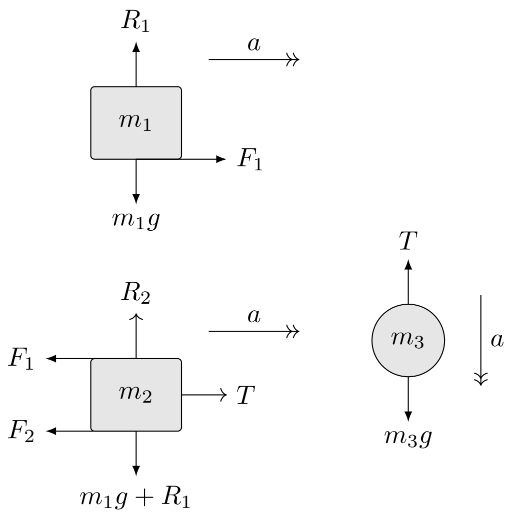
In the latter equation, using the fact that $ABCD$ is a parallelogram, we replace $|AD|$ by $|BC|$, and $\cos\angle DAB$ by $\cos(180° - \angle ABC) = -\cos\angle ABC$. This gives

$$|BD|^2 = |AB|^2 + |BC|^2 + 2|AB||BC|\cos\angle ABC.$$

Adding this to the first equation, the trigonometric terms cancel, leaving

$$|AC|^2 + |BD|^2 = 2|AB|^2 + 2|BC|^2, \text{ as required.}$$

3941. For very large m_3 compared to the coefficients of friction, the lower block will slip out from under the upper block, and the stack will not accelerate as one. In the boundary case, friction between the blocks is maximal, without slipping, and friction under the blocks is maximal, with slipping. The force diagrams, assuming rotational equilibrium, are as follows:



Consider the upper block. F_1 is maximal. So,

$$F_1 = \mu_1 R_1 = \mu_1 m_1 g.$$

Then Nɪɪ gives $\mu_1 m_1 g = m_1 a$, so $a = \mu_1 g$.

Consider the lower block. F_2 is maximal. So,

$$F_2 = \mu_2 R_2 = \mu_2(m_1 + m_2)g.$$

Taking Nɪɪ for the entire system, along the string, the tensions and internal frictions cancel.

$$\begin{aligned} & m_3 g - \mu_2(m_1+m_2)g = (m_1+m_2+m_3)\mu_1 g \\ \Longrightarrow\ & m_3(1-\mu_1) = (m_1+m_2)\mu_1 + (m_1+m_2)\mu_2 \\ \Longrightarrow\ & m_3 = \frac{(\mu_1+\mu_2)(m_1+m_2)}{1-\mu_1}. \end{aligned}$$

For m_3 larger than this value, the lower block will slip out from under the upper. Hence, since we are told that the blocks accelerate as one,

$$m_3 \leq \frac{(\mu_1+\mu_2)(m_1+m_2)}{1-\mu_1}, \text{ as required.}$$

3942. The implication is ① $\Longrightarrow$ ②.

If $\mathrm{g}(x) = \mathrm{f}(x)(ax+b)^2$, then both parts emerging from the product rule contain factors of $(ax+b)$. Hence, so does $\mathrm{g}'(x)$. But the converse is not true. Consider $\mathrm{g}(x) = (2x+3)^2 + 17$. The derivative $\mathrm{g}'(x)$ has a factor of $(2x+3)$, but $\mathrm{g}(x)$ does not have a factor of $(2x+3)^2$.

3943. By the binomial expansion,

$$\sqrt[4]{x^4 \pm h} = x\left(1 \pm \tfrac{h}{x^4}\right)^{\frac{1}{4}} = x\left(1 \pm \tfrac{h}{4x^4} + \ldots\right).$$

All other terms are in h^2 or higher. Subtracting the $\pm$ versions and taking out a factor of x gives

$$\begin{aligned} & x \lim_{h\to 0} \frac{\frac{h}{2x^4} + \text{terms in } h^2 \text{ and higher}}{2h} \\ \equiv\ & x \lim_{h\to 0} \frac{1}{4x^4} + \text{terms in } h \text{ and higher} \\ \equiv\ & \frac{1}{4x^3}. \end{aligned}$$

3944. If exactly two couples sit together, then the last couple are opposite each other. Suppose this is A_1 and A_2. There are 6 ways in which they can sit. For each of these, there are 2^3 ways of placing the other couples: 2 locations for the couples, and 2 locations for the individuals within each couple. This gives $6 \times 2^3 = 48$ ways with A_1 and A_2 apart. So, there are $3 \times 48 = 144$ ways with exactly one couple sitting apart. Out of $6! = 720$ seating plans, this gives a probability of $\frac{144}{720} = \frac{1}{5}$, as required.

— Alternative Method —

Place A_1 wlog. The probability that A_2 sits down opposite is $^1/_5$. Place B_1 wlog. The probability that B_2 sits next to B_1 is $^1/_3$. C_1 and C_2 are then guaranteed to sit together, giving success.

There are three ways in which the above can occur, with any one of the couples playing the role of A_1 and A_2 above. So, the probability p that exactly two couples sit together is

$$p = 3 \times \tfrac{1}{5} \times \tfrac{1}{3} = \tfrac{1}{5}.$$

3945. (a) The degrees are $\{5, 3, 3, 3\}$, which are odd.

(b) A visit to any land mass, except the start and endpoints of the walk, must use two of the bridges that connect it. So, for a successful walk, there can only be a maximum of two land masses with odd degree, one for the startpoint and one for the endpoint. But Königsberg has four land masses of odd degree, so such a walk is impossible. □

— Nota Bene —

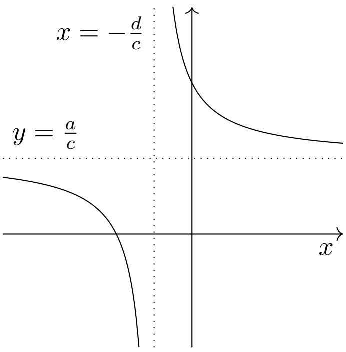
It was Leonhard Euler, one of the greatest mathematicians of all time, who provided this proof in the 18th century, thus founding the modern field of topology.

3946. (a) Rewriting the numerator in terms of $(cx + d)$.

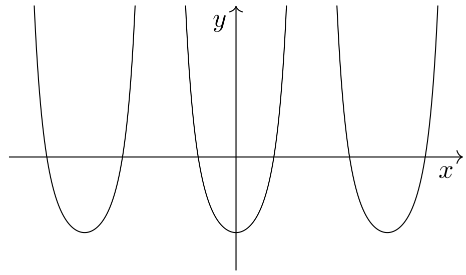
$$\begin{aligned} y &= \frac{ax+b}{cx+d} \\ &\equiv \frac{\frac{a}{c}(cx+d) - \frac{ad}{c} + b}{cx+d} \\ &\equiv \frac{a}{c} + \frac{b - \frac{ad}{c}}{cx+d} \\ &\equiv \frac{a}{c} + \frac{bc - ad}{c(cx+d)}. \end{aligned}$$

(b) i. The fraction is undefined when $cx + d = 0$, so $x = -\frac{d}{c}$ is the equation of the vertical asymptote. And when $x \to \infty$, the fraction tends to zero, so $y = \frac{a}{c}$ is the equation of the horizontal asymptote.

ii. The graph is the standard reciprocal $y = \frac{1}{x}$, stretched and translated:



3947. A counterexample is $\mathrm{f}(x) = \sec^2 x - 2$, which is convex everywhere it is defined, and has infinitely many roots:

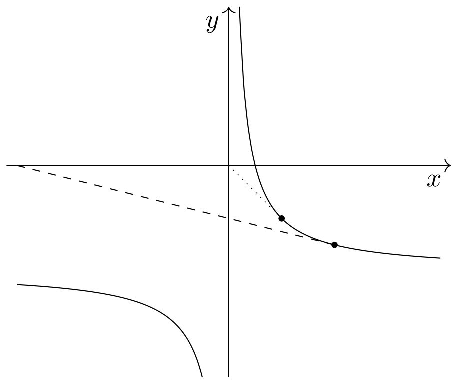


3948. (a) With $\mathrm{f}(x) = \frac{1}{x} - p$, we have $\mathrm{f}'(x) = -\frac{1}{x^2}$. So, the N-R iteration is

$$\begin{aligned} x_{n+1} &= x_n + \frac{\frac{1}{x_n} - p}{-\frac{1}{x^2}} \\ &= x_n - (x_n - px_n^2) \\ &= x_n(2 - px_n). \end{aligned}$$

(b) With $p = 3$ and $x_0 = 0.5$, we get $x_1 = 0.25$, $x_2 = 0.3125$ and then $x_n \to \frac{1}{3}$.

(c) Consider the iteration graphically:



If x_0 is large, then the tangent (dashed) will meet the x axis to the left of O. Subsequent values will diverge $x_n \to -\infty$. In the boundary case for this divergent behaviour, the tangent (dotted) passes through the origin.

So, we substitute $(0, 0)$ into the equation of the generic tangent, giving

$$\begin{aligned} \tfrac{2}{x_0} - p &= 0 \\ \Longrightarrow x_0 &= \tfrac{2}{p}. \end{aligned}$$

The boundary case itself also fails, since x_1 is undefined. Hence, for $x_0 \geq {}^2/_p$, the iteration will fail to converge to the reciprocal.

3949. Adding the first and third equations, we have two equations in x and y:

$$\begin{aligned} 3x + 2y &= 7 \\ 6x + 4y &= 3. \end{aligned}$$

Subtracting the second equation from two copies of the first equation gives $0 = 11$. Hence, there are no simultaneous solutions, as required.

3950. This is true. If $x = \alpha$ is an x intercept of $y = \text{f}(x)$, then $\text{f}(\alpha) = 0$. So, $(\alpha, 0)$ also satisfies the equation $y^2 = \text{f}(x)$.

3951. (a) The curve is a quadratic in y:

$$y^2 + x^3 y - 1 = 0$$
$$\Longrightarrow y = \frac{-x^3 \pm \sqrt{x^6 + 4}}{2}.$$

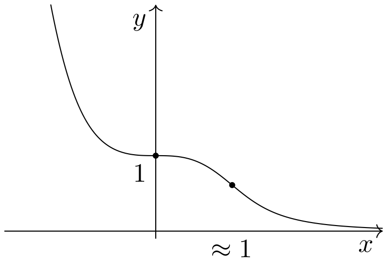
Since $\sqrt{x^6 + 4} > x^3$, one root is positive and the other negative. We are told that $y > 0$, so the equation of the curve may be written

$$2y = -x^3 + \sqrt{x^6 + 4}.$$

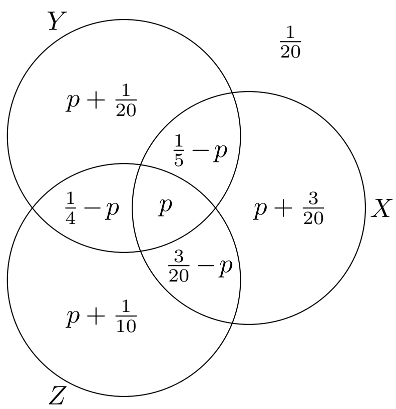
(b) As x get very large, 4 is negligible compared to x^6, so $\sqrt{x^6 + 4} \to x^3$. Hence, $y \to 0^+$.

(c) As the curve tends towards $y = 0^+$, it must be convex. And we are told that there are precisely two points of inflection, at which the second derivative is zero and changes sign. So, C must be convex on $(-\infty, 0)$, concave on $(0, \approx 1)$ and convex on $(\approx 1, \infty)$.

(d) As $x \to -\infty$, $y \to \infty$. Altogether, this is



3952. Let $p = \mathbb{P}(X \cap Y \cap Z)$. Then, we work from the inside of the Venn diagram outwards. Since $\mathbb{P}(X \cap Y) = \frac{1}{5}$, we know $\mathbb{P}(X \cap Y \cap Z') = \frac{1}{5} - p$. Continuing in this way, we fill the Venn diagram as follows:



Equating the sum of all the probabilities to 1,

$$\tfrac{1}{20} + \tfrac{3}{20} + \tfrac{1}{10} + \tfrac{1}{5} + \tfrac{1}{4} + \tfrac{3}{20} + p + \tfrac{1}{20} = 1$$
$$\Longrightarrow p = \tfrac{1}{20}.$$

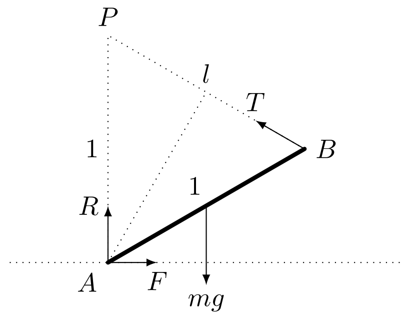
3953. Setting the first derivative to zero,

$$3\sin^2 x \cos x - 3\cos^2 x \sin x = 0$$
$$\Longrightarrow \sin x \cos x (\sin x - \cos x) = 0$$
$$\Longrightarrow \sin x = 0, \text{ or } \cos x = 0, \text{ or } \tan x = 1.$$

In $[0, 2\pi)$, this gives six stationary points. Their coordinates are

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
y	1	$\frac{\sqrt{2}}{2}$	1	-1	$-\frac{\sqrt{2}}{2}$	-1

3954. The force diagram is



$\triangle BAP$ is isosceles. So, let $\angle BAP = 2\theta$, which gives $\sin\theta = {}^{l}\!/_{2}$ and $\cos\theta = \sqrt{1 - {}^{l^2}\!/_{4}}$. The angle of inclination of the tension is also θ.

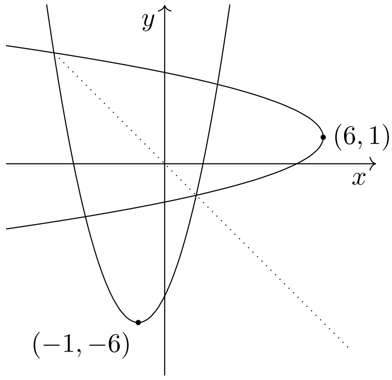
Resolving vertically,

$$T\sin\theta + R - mg = 0$$
$$\Longrightarrow R = mg - \tfrac{1}{2}mgl \cdot \tfrac{1}{2}l$$
$$= mg\big(1 - \tfrac{1}{4}l^2\big).$$

Assuming that friction at the ground is limiting, $F = \mu R$. Horizontally, $\mu R = T\cos\theta$. Substituting values in,

$$\mu mg\big(1 - \tfrac{1}{4}l^2\big) \geq \tfrac{1}{2}mgl\sqrt{1 - \tfrac{1}{4}l^2}$$
$$\Longrightarrow \mu \geq \frac{l}{2\sqrt{1 - \frac{1}{4}l^2}}$$
$$\equiv \frac{l}{\sqrt{4 - l^2}}, \text{ as required.}$$

3955. Completing the square to $y = (x+1)^2 - 6$, the old vertex is at $(-1, -6)$. Reflecting this point in $y = -x$, the new vertex is at $(6, 1)$:



So, the new equation is $x = -(y-1)^2 + 6$. When expanded, this is $x = -y^2 + 2y + 5$.

Alternative Method

Reflection in the line $y = -x$ is replacement of x by $-y$ and y by $-x$. Substituting this into the original equation,

$$\begin{aligned}(-x) &= (-y)^2 + 2(-y) - 5\\ \Longrightarrow -x &= y^2 - 2y - 5\\ \Longrightarrow x &= -y^2 + 2y + 5.\end{aligned}$$

3956. Either $x = 0$, or $x^3 + x^2 - 2 = 0$. This has a root at $x = 1$, so $(x-1)$ is a factor. Taking this out, the full equation is $x(x-1)(x^2+2x+2) = 0$. The quadratic factor has $\Delta = -4 < 0$, so has no real roots. Hence, the solution is $x \in \{0, 1\}$.

3957. A holiday house! (Log cabin plus sea.)

3958. The derivatives are

$$\begin{aligned}y &= ax^3 + bx^2 + cx + d\\ \Longrightarrow \tfrac{dy}{dx} &= 3ax^2 + 2bx + c\\ \Longrightarrow \tfrac{d^2y}{dx^2} &\equiv 6ax + 2b = 2(3ax + b).\end{aligned}$$

① Setting the first derivative to zero for SPs,

$$\begin{aligned}3ax^2 + 2bx + c &= 0\\ \Longrightarrow x &= \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a}\\ &\equiv \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}.\end{aligned}$$

Since $b^2 - 3ac > 0$, this has distinct real roots. These are either side of $x = \frac{-b}{3a}$.

② The second derivative has a single factor $(3ax + b)$, so it has a single root at $x = -\frac{b}{3a}$. At a single root, there is a sign change. So, the graph has a point of inflection at $x = -\frac{b}{3a}$.

The results are the same, as required.

Nota Bene

This proves that, where a cubic has two stationary points, the stationary points are equidistant from the point of inflection.

3959. (a) Equating the x coordinates, $5t = 120 - 3t$, which gives $t = 15$. Substituting this into the y coordinates, $\mathbf{r}_a = 80 - 2 \cdot 15 = 50$ and $\mathbf{r}_b = 110 - 4 \cdot 15 = 50$. Hence, the boats collide.

(b) At $t = 0$, the distance between the boats is given by Pythagoras as $\sqrt{115^2 + 30^2}$. Since they are travelling at constant velocity, this distance is halved when they are halfway to collision, giving 59.4 nautical miles.

3960. (a) The relevant version of the cosine rule is

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}.$$

Differentiating implicitly, (noting that a and c may be treated as constants),

$$-\sin B \frac{dB}{dt} = \frac{-2b\frac{db}{dt}}{2ac}.$$

Substituting values into the RHS,

$$\sin B \frac{dB}{dt} = \sqrt{3}, \text{ as required.}$$

(b) The triangle is right-angled at this point. So, $\sin B = \sqrt{3}/2$. Substituting into part (a),

$$\begin{aligned}\frac{\sqrt{3}}{2} \cdot \frac{dB}{dt} &= \sqrt{3}\\ \Longrightarrow \frac{dB}{dt} &= 2 \text{ radians per second.}\end{aligned}$$

3961. (a) With $b = 0$, the iteration is

$$\begin{aligned}x_{n+1} &= x_n^2 - y_n^2 + a,\\ y_{n+1} &= 2x_n y_n.\end{aligned}$$

This produces $x_1 = a$ and $y_1 = 0$. Then $x_2 = a^2 + a$.

(b) With $a = 0$, the iteration is

$$\begin{aligned}x_{n+1} &= x_n^2 - y_n^2,\\ y_{n+1} &= 2x_n y_n + b.\end{aligned}$$

This produces $x_1 = 0$, $y_1 = b$. Then $x_2 = -b^2$, $y_2 = b$. Then $x_3 = b^4 - b^2$, $y_3 = -2b^3 + b$. So, the equation $x_3 = y_3 + y_2$ is

$$\begin{aligned}b^4 - b^2 &= -2b^3 + b + b\\ \Longrightarrow b(b-1)(b+1)(b+2) &= 0\\ \Longrightarrow b &= 0, 1, -1, -2.\end{aligned}$$

3962. (a) The derivative is $3x^2$, so the normal gradient at $\big(p, p^3\big)$ is $\frac{-1}{3p^2}$. Hence, the equation of the tangent is

$$\begin{aligned}y - p^3 &= \tfrac{-1}{3p^2}(x - p)\\ \Longrightarrow 3p^2 y &= -x + p + 3p^5\\ \Longrightarrow 3p^2 y + x - p - 3p^5 &= 0.\end{aligned}$$

(b) The shortest path to the cubic lies along the normal. Substituting $(14, 7)$ into the equation of the normal, $21p^2 + 14 - p - 3p^5 = 0$. This is a quintic, so we cannot use a polynomial solver. The N-R iteration is

$$p_{n+1} = p_n - \frac{21p_n^2 + 14 - p_n - 3p_n^5}{42p_n - 1 - 15p_n^4}.$$

Running this with $p_0 = 0$ gives $p_n \to 2$. So, the closest point to $(14, 7)$ is $(2, 8)$, at a distance of $\sqrt{145}$.

Substituting instead $(13, -3)$, the quintic is $-9p^2 + 13 - p - 3p^5 = 0$. The N-R iteration is

$$p_{n+1} = p_n - \frac{-9p_n^2 + 13 - p_n - 3p_n^5}{-18p_n - 1 - 15p_n^4}.$$

Running this with $p_0 = 0$, we get $p_n \to 1$. So, the closest point to $(13, -3)$ is $(1, 1)$, lying at a distance $\sqrt{160}$.

So, the point $(13, -3)$ is further from $y = x^3$.

3963. (a) The components of velocity are $v_x = u\cos\theta$ and $v_y = u\sin\theta$. So, the displacements are

$$\begin{aligned} x &= (u\cos\theta)t \\ y &= (u\sin\theta)t - \tfrac{1}{2}gt^2. \end{aligned}$$

(b) Setting $y = 0$, we solve for t, rejecting $t = 0$. The time of flight is

$$t = \frac{2u\sin\theta}{g}.$$

Substituting into the horizontal, the range is

$$R = \frac{2u^2\sin\theta\cos\theta}{g}.$$

(c) Using a double-angle formula, the range is

$$R = \frac{u\sin 2\theta}{g}.$$

This is maximised when $\sin 2\theta = 1$, which is at $\theta = 45°$, as required.

3964. (a) Equating terms in x,

$$\begin{aligned} 12960x &\equiv {}^b\text{C}_1(2x)^1a^{b-1} \\ \Longrightarrow 6480 &= ba^{b-1}. \end{aligned}$$

Equating terms in x^2,

$$\begin{aligned} 8640x^2 &\equiv {}^b\text{C}_2(2x)^2a^{b-2} \\ \Longrightarrow 2160 &= \tfrac{b!}{2!(b-2)!}a^{b-2} \\ \Longrightarrow 4320 &= b(b-1)a^{b-2}. \end{aligned}$$

(b) Dividing the first equation by the second,

$$\begin{aligned} \frac{6480}{4320} &= \frac{a}{b-1} \\ \Longrightarrow a &= \tfrac{3}{2}(b-1). \end{aligned}$$

Since $a \in \mathbb{N}$, $(b-1)$ must have a factor of 2.

(c) Using the given prime factorisation,

$$2^4 \cdot 3^4 \cdot 5 = ba^{b-1}.$$

Since $b - 1$ is even, a^{b-1} cannot contain the single factor of 5. So, b contains the factor of 5. And we know that $b < 10$. So, $b = 5$ and $a = 6$.

3965. Setting $y = 0$, the x intercept is at $x = c$. So, the relevant integral is

$$\begin{aligned} A &= \int_0^c \left(\sqrt{c} - \sqrt{x}\right)^2 \, dx \\ &\equiv \int_0^c c - 2\sqrt{c}\sqrt{x} + x \, dx \\ &\equiv \left[cx - \tfrac{4}{3}\sqrt{c}x^{\frac{3}{2}} + \tfrac{1}{2}x^2\right]_0^c \\ &\equiv c^2 - \tfrac{4}{3}c^2 + \tfrac{1}{2}c^2 \\ &\equiv \tfrac{1}{6}c^2. \end{aligned}$$

3966. (a) The largest real domain of h cannot include the root of the denominator. So, it is $\mathbb{R} \setminus \{1/2\}$.

(b) We write the numerator as

$$16x^4 \equiv (2x-1)(ax^3 + bx^2 + cx + d) + e.$$

Equating coefficients, starting with x^4, $a = 8$, then $b = 4$, then $c = 2$, then $d = 1$, leaving $e = 1$. This gives

$$\begin{aligned} &\frac{16x^4}{2x-1} \\ &\equiv \frac{(2x-1)(8x^3 + 4x^2 + 2x + 1) + 1}{2x-1} \\ &\equiv 8x^3 + 4x^2 + 2x + 1 + \frac{1}{2x-1}. \end{aligned}$$

—— Alternative Method ——

Using polynomial long division,

$$\begin{array}{r l} & 8x^3 + 4x^2 + 2x + 1 \\ 2x - 1 \big) & 16x^4 + 0x^3 + 0x^2 + 0x + 0 \\ & -\,16x^4 + 8x^3 \\ & 8x^3 + 0x^2 \\ & -\,8x^3 + 4x^2 \\ & 4x^2 + 0x \\ & -\,4x^2 + 2x \\ & 2x + 0 \\ & -\,2x + 1 \\ & 1 \end{array}$$

So, the quotient is $8x^3 + 4x^2 + 2x + 1$ and the remainder is 1, which gives

$$8x^3 + 4x^2 + 2x + 1 + \frac{1}{2x-1}.$$

(c) The graph $y = \text{h}(x)$ is a positive cubic plus a reciprocal. So, when not in the vicinity of $x = 1/2$, the curve resembles a positive cubic. Particularly, as $x \to \pm\infty$, $y \to \pm\infty$. Hence, the range is $(-\infty, 0] \cup [256/27, \infty)$.

3967. We use the fact that $\log_p q \equiv \frac{1}{\log_q p}$. Starting with the RHS,

$$\begin{aligned}&\frac{1}{\frac{1}{\log_x a}+\frac{1}{\log_y a}}\\ \equiv\ &\frac{1}{\log_a x+\log_a y}\\ \equiv\ &\frac{1}{\log_a xy}\\ \equiv\ &\log_{xy} a, \text{ as required.}\end{aligned}$$

3968. The range of both $\sin t$ and $\sin kt$ is $[-1,1]$. So, since $(1,1,\sqrt{2})$ is a right-angled triangle, if the particle is $\sqrt{2}$ away from the origin, each of $\sin t$ and $\sin kt$ must equal ± 1. For $\sin t$, this occurs at

$$t=\tfrac{\pi}{2},\tfrac{3\pi}{2},\tfrac{5\pi}{2},\tfrac{7\pi}{2},....$$

(a) Evaluating $\sin 2t$ at these times, we get 0 in each case. Hence, the x and y coordinates never both have value ± 1, and the particle is never $\sqrt{2}$ units from the origin.

(b) Evaluating $\sin 3t$, we get $-1,1,-1,1,...$ Hence, whenever the x position satisfies $|x|=1$, the y position satisfies $|y|=1$. Hence, the particle is $\sqrt{2}$ units from the origin at $t=\frac{\pi}{2},\frac{3\pi}{2},....$

3969. Squaring both equations,

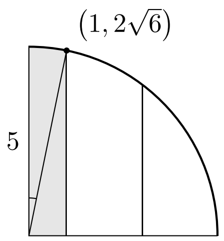
$$\begin{aligned}R^2\cot^2\theta&=4,\\ R^2\operatorname{cosec}^2\theta&=9.\end{aligned}$$

The third Pythagorean identity, scaled by R^2, is

$$R^2\cot^2\theta+R^2\equiv R^2\operatorname{cosec}^2\theta.$$

Substituting in, $4+R^2=9$, so $R=\pm\sqrt{5}$.

3970. The equation of the circle is $x^2+y^2=25$, and the equations of the sides of the stripe are $x=\pm 1$. Consider only the positive quadrant:



The shaded region consists of a triangle of area $\sqrt{6}$, and a sector. The sector subtends an angle θ at the centre, where

$$\tan\theta=\frac{1}{2\sqrt{6}}.$$

So, the area of the sector is $\frac{1}{2}\theta=0.100678....$ This gives the area of the shaded region as

$$\sqrt{6}+0.1006...=2.5501....$$

To find the area of the central stripe, we multiply by 4, giving 10.200..., so 10.2 (3sf).

3971. (a) i. Over the interval δt, change in velocity is 1.4 ms^{-1}, so the acceleration of each tablet is $\frac{1.4}{\delta t}$ ms^{-2},

ii. Since 2.4 kg emerges per second, $2.4\delta t$ kg emerges during an interval of length δt.

(b) Over the interval δt, NII is

$$F=ma=2.4\delta t\times\frac{1.4}{\delta t}.$$

This gives $F=3.36$ N, which is independent of the interval δt.

3972. (a) If $a<b$, then the sequence proceeds

$$\begin{aligned}P_1&=a,\\ P_2&=|a|-b=a-b\\ P_3&=|a-b|-b=-(a-b)-b=-a\\ P_4&=|-a|-b=a-b.\end{aligned}$$

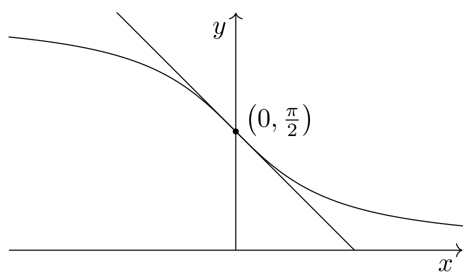
Since $P_4=P_2$, we know that $P_5=P_3$ and so forth. The sequence is periodic after the first term.

(b) If $b\le a<2b$, then the sequence proceeds

$$\begin{aligned}P_1&=a,\\ P_2&=|a|-b=a-b\\ P_3&=|a-b|-b=(a-b)-b=a-2b\\ P_4&=|a-2b|-b=-(a-2b)-b=-a+b\\ P_5&=|-a+b|-b=-(-a+b)-b=a-2b.\end{aligned}$$

Since $P_5=P_3$, we know that $P_6=P_4$ and so forth. The sequence is periodic after the second term.

3973. The graphs of $y=\operatorname{arccot} x$ and $y=\frac{\pi}{2}-x$ are as follows (we are not yet assuming proof that they intersect at only one point):



The point $x=0$, $y=\frac{\pi}{2}$ satisfies both equations. The gradient of the straight line is -1. To find the gradient of the arccot graph, we rearrange to $x=\cot y$ and differentiate with respect to x:

$$\begin{aligned}1&=-\operatorname{cosec}^2 y\cdot\frac{dy}{dx}\\ \Longrightarrow\frac{dy}{dx}&=-\sin^2 y.\end{aligned}$$

So, the gradient of $y = \operatorname{arccot} x$ takes values in $[-1, 1]$. It is only equal to -1 at $(0, \pi/2)$. So, $y = \operatorname{arccot} x$ is shallower than $y = \pi/2 - x$ at every point other than their point of intersection. This proves that the graphs do not intersect again, as shown.

3974. Differentiating, $f'(x) = 3x^2 - 4 + \frac{1}{x}$. For SPs,

$$\begin{aligned} &3x^2 - 4 + \tfrac{1}{x} = 0 \\ \Longrightarrow\ &3x^3 - 4x + 1 = 0. \end{aligned}$$

This has a root at $x = 1$. The second derivative is

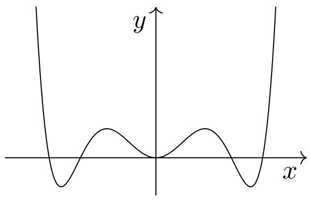
$$f''(x) = 6x - \frac{1}{x^2}.$$

So, $f''(1) = 5$, meaning that the SP is a turning point (a minimum). The function isn't one-to-one over $\mathbb{R}^+$, and is therefore not invertible.

3975. There is a common factor of $x^2 - 1 \equiv (x+1)(x-1)$. Taking this out, we have $y = (x^2-1)\big((x^2-1)^2-1\big)$. The latter factor is a difference of two squares:

$$\begin{aligned} y &= (x^2 - 1)\big((x^2 - 1)^2 - 1\big) \\ &\equiv (x^2 - 1)\big((x^2 - 1) - 1\big)\big((x^2 - 1) + 1\big) \\ &\equiv x^2(x^2 - 1)(x^2 - 2) \\ &\equiv x^2(x + 1)(x - 1)\big(x + \sqrt{2}\big)\big(x - \sqrt{2}\big). \end{aligned}$$

So, the graph is a positive sextic with a double root at $x = 0$ and four single roots at $x = \pm 1, \pm\sqrt{2}$:



3976. The distribution of the sample means is

$$\bar{X} \sim N\left(215, \frac{16^2}{n}\right).$$

We are looking for

$$\mathbb{P}(|\bar{X} - 215| > 10) = 0.01.$$

By symmetry, this is

$$\mathbb{P}(\bar{X} - 215 > 10) = 0.005.$$

Using the inverse normal distribution, the z-value is 2.576. So, we require

$$\begin{aligned} &\frac{10}{\frac{16}{\sqrt{n}}} = 2.576 \\ \Longrightarrow\ &\sqrt{n} = \frac{2.576 \times 16}{10} \\ \Longrightarrow\ &n = 16.99 \text{ (4sf)}. \end{aligned}$$

Hence, the least value of n is 17.

3977. (a) The parabola remains monic under reflection in $x = p$. The original roots were a and b, which can be thought of as $a - p$ and $b - p$ to the right of $x = p$. So, the new roots are $a - p$ and $b - p$ to the left of $x = p$:

$$\begin{aligned} p - (a - p) &\equiv 2p - a, \\ p - (b - p) &= 2p - b. \end{aligned}$$

Hence, the equation of the new parabola is

$$y = (x - 2p + a)(x - 2p + b).$$

(b) Reflection in the line $y = q$ can be considered as reflection in $y = 0$ followed by translation by vector $2q\mathbf{j}$. Enacting these in order, the equation of the new parabola is

$$y = -(x - a)(x - b) + 2q.$$

3978. (a) Differentiating numerator and denominator, L'Hôpital's rule tells us that

$$\lim_{k\to 0} \frac{\ln k}{\frac{1}{k}} = \lim_{k\to 0} \frac{\frac{1}{k}}{-\frac{1}{k^2}} = \lim_{k\to 0} -k = 0.$$

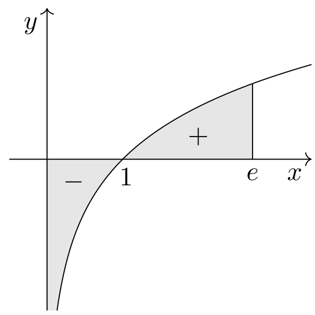
(b) We integrate by parts. Let $u = \ln x$ and $v' = 1$. So $u' = 1/x$ and $v = x$. This gives

$$\begin{aligned} \int \ln x\, dx &= x \ln x - \int 1\, dx \\ &= x \ln x - x + c. \end{aligned}$$

We can now calculate

$$\begin{aligned} I &= \lim_{k\to 0} \int_k^e \ln x\, dx \\ &= \lim_{k\to 0} \Big[x \ln x - x\Big]_k^e \\ &= \lim_{k\to 0} \big((0) - (k \ln k - k)\big) \\ &= 0. \end{aligned}$$

(c) The region between the curve and the x axis for $x \in (0, 1]$, despite being infinite in extent, is bounded in area. Its area is equal to that of the region between the curve and the x axis for $x \in [1, e]$. Since these regions are on opposite sides of the x axis, their signed areas cancel, giving $I = 0$:



3979. The components of velocity are $\dot{x} = 2t$ and $\dot{y} = 4t^3$. For the particle to be moving parallel to the vector $\mathbf{i} + 2\mathbf{j}$, we require

$$\begin{aligned} \dot{y} &= 2\dot{x} \\ \implies 4t^3 &= 4t \\ \implies t &= 0, \pm 1. \end{aligned}$$

At these points, the velocities and coordinates are

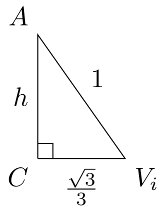
t	$\mathbf{v}$	(x, y)
-1	$-2\mathbf{i} - 4\mathbf{j}$	$(1, 0)$
0	$0\mathbf{i} + 0\mathbf{j}$	$(0, -1)$
1	$2\mathbf{i} + 4\mathbf{j}$	$(1, 0)$

At $t = -1$, the velocity is in the opposite direction to $\mathbf{i} + 2\mathbf{j}$. At $t = 0$, the particle is at rest. Only at $t = 1$ is the particle moving in the same direction as $\mathbf{i} + 2\mathbf{j}$. At this time, it is at $(1, 0)$.

— Nota Bene —

You could reasonably interpret this question so as to allow $t = -1$. This gives the same point $(1, 0)$, so doesn't change the answer. The point $(0, -1)$, however, should certainly not be included.

3980. Initially, set $l = 1$. The centre of a triangle divides its medians in the ratio $1 : 2$. The tetrahedron's base is an equilateral triangle, whose (horizontal) height is $\sqrt{3}/2$, so the centre C of the base lies $\sqrt{3}/3$ away from its vertices V_i. This gives the following vertical triangle, with apex A:



By Pythagoras, $h^2 = 1 - \frac{1}{3} = \frac{2}{3}$. So, $h = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$. Scaling the lengths up by l gives

$$h = \frac{\sqrt{6}}{3} l, \text{ as required.}$$

3981. We are looking for the probability that the largest of three dice rolls Z exceeds the sum of the other two $(X + Y)$ by at least three. The possibility space contains $6^3 = 216$ outcomes. Of these, the successful outcomes are

$$\begin{aligned} (1, 1, 5) &: \text{ 3 orders,} \\ (1, 1, 6) &: \text{ 3 orders,} \\ (1, 2, 6) &: \text{ 6 orders.} \end{aligned}$$

So, the probability is $\frac{12}{216} = \frac{1}{18}$.

3982. (a) Setting the denominator to zero,

$$\begin{aligned} 42x^2 - 71x + 30 &= 0 \\ \implies x &= \tfrac{5}{6}, \tfrac{6}{7}. \end{aligned}$$

These are the vertical asymptotes. There is also a horizontal asymptote: as $x \to \pm\infty$, the fraction tends to zero, leaving $y = 169$ as the horizontal asymptote.

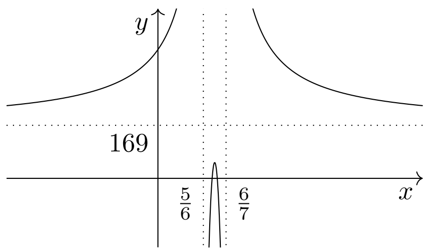
(b) A quadratic has a line of symmetry. Hence, $y = \mathrm{f}(x)$ does too. The curve is stationary at this x value. It is midway between the roots, at $x = \frac{1}{2}\left(\frac{5}{6} + \frac{6}{7}\right) = \frac{71}{84}$, as required.

(c) The range of the denominator is $[-1/168, \infty)$. Reciprocating this, the range of the fraction is $(-\infty, -168) \cup (0, \infty)$. This gives the range of the function as $(-\infty, 1) \cup (169, \infty)$.

(d) Setting $\mathrm{f}(x)$ to zero, the roots are at

$$\begin{aligned} \frac{1}{42x^2 - 71x + 30} + 169 &= 0 \\ \implies 169(42x^2 - 71x + 30) - 1 &= 0 \\ \implies x &= 0.828, 0.862 \text{ (3sf).} \end{aligned}$$

These values are between the asymptotes, and either side of the stationary point. The curve, not to scale, has the following behaviour:



Choosing any starting value x_0 not between the vertical asymptotes will cause divergence, as the upper sections of the graph lead down away from the roots. Only $x_0 \in \left(5/6, 6/7\right)$ will converge. But no integers lie in this interval.

3983. The lamina is rotationally symmetrical. This tells us that the magnitude R_{outer} of the reaction forces at the three outer supports must be the same. It tells us nothing about the magnitude R_{inner} of the force at the central support.

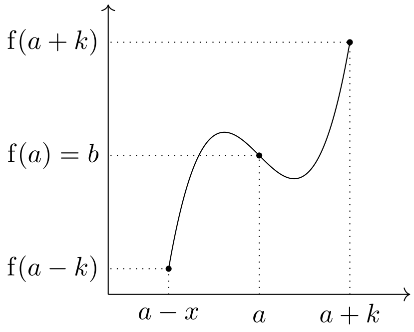
Consider the following possibilities:

① $R_{\text{outer}} = \frac{1}{3}mg, \quad R_{\text{inner}} = 0,$

② $R_{\text{outer}} = 0, \quad R_{\text{inner}} = mg.$

In both of these cases, the lamina is in equilibrium (albeit unstable equilibrium in the second case). In reality, the forces will be somewhere in between these two extremes, spread between the edges and the centre. But it is not possible to tell (without detailed modelling of the lamina itself, for which we have no information) exactly how the load is shared.

3984. If $y = \mathrm{f}(x)$ has rotational symmetry around the point (a, b), then the x coordinates $a+k$ and $a-k$ (symmetrically either side of $x = a$) must produce y values symmetrically above and below $y = b$:



Algebraically, this may be expressed as

$$\mathrm{f}(a+k) - b = b - \mathrm{f}(a-k).$$

Differentiating by the chain rule,

$$\begin{aligned}\mathrm{f}'(a+k) &= -(-\,\mathrm{f}'(a-k)) \\ &= \mathrm{f}'(a-k).\end{aligned}$$

So, at values symmetrical in $x = a$, the gradients are equal. This implies that $y = \mathrm{f}'(x)$ has $x = a$ as a line of symmetry. $\square$

3985. (a) This is true. If $k < 0$, then the first probability is 1, which is larger than the second. If $k \geq 0$, then the first probability may be rewritten as

$$\begin{aligned}\mathbb{P}(|Z| > k) &= \mathbb{P}(Z < -k \text{ or } Z > k) \\ &= \mathbb{P}(Z < -k) + \mathbb{P}(Z > k) \\ &> \mathbb{P}(Z > k).\end{aligned}$$

(b) This is also true. Algebraically,

$$\begin{aligned}\mathbb{P}(Z > k) &= \mathbb{P}(k < Z \leq k+1) \\ &\quad + \mathbb{P}(Z > k+1) \\ \Longrightarrow \mathbb{P}(Z > k+1) &< \mathbb{P}(Z > k).\end{aligned}$$

3986. (a) At the changeover between the definitions, $\mathrm{f}(2) = 2$. So, the limit of the first definition must approach 2 as $x \to 2$. This amounts to substituting $x = 2$ into the first definition, which gives $k - 6 = 2 \implies k = 8$.

(b) The value of $\mathrm{f}'(x)$ in the first definition is -3. The derivative in the second definition is given, using the quotient rule, by

$$\begin{aligned}\mathrm{f}'(x) &= \frac{(x-1)^2 - 2x(x-1)}{(x-1)^4} \\ &\equiv \frac{-x-1}{(x-1)^3}.\end{aligned}$$

This gives $\mathrm{f}'(2) = \frac{-3}{1} = -3$. So, the values of $\mathrm{f}'(x)$ match, and there is no discontinuity in f', other than at $x = 1$.

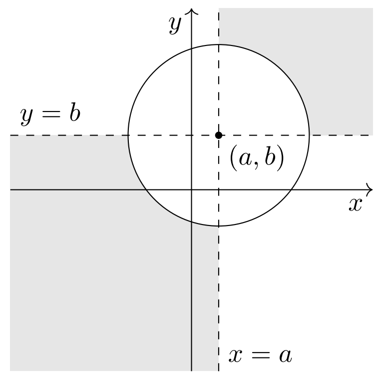
3987. (a) True.

(b) False. For a counterexample, at $x = 2$, the LHS has value 63, while the RHS has value 45.

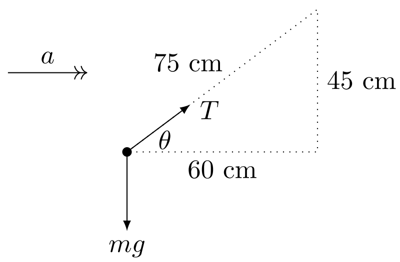
(c) True.

3988. The first inequality is satisfied if the factors $(x-a)$ and $(y-b)$ are both positive or both negative. This gives two regions bounded by $x = a$ and $y = b$, to the upper right and lower left.

The second inequality requires the distance from (a, b) to be greater than 1. This defines the region outside a circle:



3989. In the boundary case, the reaction force is zero, but the cans are still at ground level. In this case, the forces on a can are



The angle of inclination of the string is θ, where

$$\begin{aligned}\sin\theta &= \tfrac{45}{75} = \tfrac{3}{5}, \\ \cos\theta &= \tfrac{60}{75} = \tfrac{4}{5}.\end{aligned}$$

Resolving vertically, $T\sin\theta = mg$, so $T = \frac{5}{3}mg$. Horizontally,

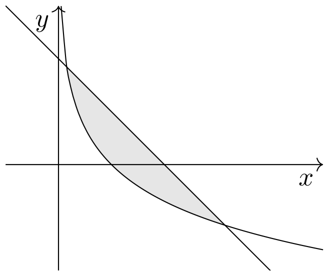
$$\begin{aligned}T\cos\theta &= ma \\ \Longrightarrow a &= \tfrac{5}{3}g \cdot \tfrac{5}{4} \\ &= \tfrac{4}{3}g \text{ ms}^{-2}.\end{aligned}$$

— Nota Bene —

Clearly, the assumption of smoothness here bears no relation to reality! But the answer, which is exceedingly large for a car (greater than freefall acceleration!) does broadly correspond to reality. It would require an unfeasibly large acceleration to *keep* the cans in the air.

3990. Every cubic is rotationally symmetrical around its point of inflection. Hence, the transformation has no effect, giving $y = 4x^3 - 6x - 7$.

3991. (a) The boundary equations are $x = e^{-y}$, which is $y = -\ln x$, and the straight line $x + y = 2$:



(b) For intersections, $x - \ln x - 2 = 0$. This is not analytically solvable. The N-R iteration is

$$x_{n+1} = x_n - \frac{x_n - \ln x_n - 2}{1 - \frac{1}{x_n}}.$$

With $x_0 = 0.1$, $x_n \to 0.15859$. With $x_0 = 2$, $x_n \to 3.14619$.

(c) The area enclosed by the two curves is

$$\begin{aligned} A &= \int_{0.15859}^{3.14619} 2 - x + \ln x \, dx \\ &= \Big[2x - \tfrac{1}{2}x^2 + x\ln x - x\Big]_{0.15859}^{3.14619} \\ &= (1.8030...) - (-0.14601...) \\ &= 1.949 \text{ (4sf)}. \end{aligned}$$

3992. Consider the equation $f(x) = 2x$:

$$\begin{aligned} & x^4 - x^2 + 2x + 1 = 2x \\ \Longrightarrow\ & x^4 - x^2 + 1 = 0. \end{aligned}$$

This is a biquadratic with $\Delta = -3$. So, $f(x) = 2x$ has no roots. Consider the curve $y = f(x)$. This is a positive quartic. It doesn't intersect $y = 2x$, so must be above the line $y = 2x$ everywhere. Hence, $f(x) > 2x$, proving the result.

3993. (a) The gradient of AP is

$$\frac{\Delta y}{\Delta x} = \frac{\sin\theta}{\cos\theta + 1}.$$

(b) The gradient of BP is $\dfrac{-\sin\theta}{1 - \cos\theta}$.

(c) The product of the two gradients is

$$\begin{aligned} & \frac{\sin\theta}{\cos\theta + 1} \times \frac{-\sin\theta}{1 - \cos\theta} \\ \equiv\ & \frac{-\sin^2\theta}{1 - \cos^2\theta} \\ \equiv\ & \frac{-\sin^2\theta}{\sin^2\theta} \\ \equiv\ & -1. \end{aligned}$$

Hence, the angle in a semicircle is 90°. QED.

3994. Separating the variables,

$$\begin{aligned} & y' \cot x = y \\ \Longrightarrow\ & \int \frac{1}{y}\, dy = \int \tan x \, dx \\ \Longrightarrow\ & \ln|y| = \ln|\sec x| + c \\ \therefore\ & y = A\sec x. \end{aligned}$$

Substituting $x = 0$, $y = 2$, the particular solution is $y = 2\sec x$.

3995. Assume, for a contradiction, that there are $a, b \in \mathbb{N}$ such that $ax + by = 1$, and also that x and y have a common factor $k > 1$.

We can express $x = kp$ and $y = kq$, for $p, q \in \mathbb{N}$. This gives $akp + bkq = 1$. Dividing through by k,

$$ap + bq = \tfrac{1}{k}.$$

The LHS is an integer, while the RHS is not. This is a contradiction. Hence, if there exist $a, b \in \mathbb{N}$ such that $ax + by = 1$, then x and y have no common factor other than 1. □

3996. (a) Let $y = g(x)$. Separating the variables,

$$\begin{aligned} & \frac{dy}{dx} = y + y^2 \\ \Longrightarrow\ & \int \frac{1}{y(y+1)}\, dy = \int 1 \, dx. \end{aligned}$$

Writing in partial fractions,

$$\begin{aligned} & \int \frac{1}{y} - \frac{1}{y+1}\, dy = x \\ \Longrightarrow\ & \ln|y| - \ln|y+1| = x + c \\ \Longrightarrow\ & \ln\left|\frac{y}{y+1}\right| = x + c \\ \Longrightarrow\ & \frac{y}{y+1} = Ae^x. \end{aligned}$$

Substituting the conditions $x = 0$, $y = 1$ gives $A = 1/2$. Rearranging to make y the subject,

$$\begin{aligned} & 2y = e^x y + e^x \\ \Longrightarrow\ & y(2 - e^x) = e^x \\ \Longrightarrow\ & y = \frac{e^x}{2 - e^x}. \end{aligned}$$

(b) Substituting back into the original DE,

$$\begin{aligned} g'(x) &= \frac{e^x}{2 - e^x} + \frac{e^{2x}}{(2 - e^x)^2} \\ &\equiv \frac{e^x(2 - e^x) + e^{2x}}{(2 - e^x)^2} \\ &\equiv \frac{2e^x}{(2 - e^x)^2}. \end{aligned}$$

Both numerator and denominator are positive, so the function g is always increasing.

3997. Since **a** and **b** are perpendicular, we can proceed as if they were **i** and **j**. Equating coefficients of **a**,

$$\begin{aligned} t^2 &= 2t - 1 \\ \Longrightarrow t &= 1. \end{aligned}$$

Substituting $t = 1$ into the coefficients of **b**, we see that, at $t = 1$, both particles are at position vector $\mathbf{a} + \mathbf{b}$. Hence, they collide, as required.

——— Nota Bene ———

The logic runs equivalently using **a** and **b**, which we are *told* are perpendicular unit vectors, as it would using the standard **i** and **j**, which are *known* to be perpendicular unit vectors. The point being that global rotation of the problem, such as would map **a** and **b** onto **i** and **j**, makes no difference to the argument.

3998. The derivative is

$$\frac{dy}{dx} = \tfrac{1}{3}x^{-\frac{2}{3}}.$$

So, the tangent line has gradient $\frac{1}{3}a^{-\frac{2}{5}}$ and passes through $\left(a, a^{\frac{1}{3}}\right)$ and $(-2/5, 0)$. Substituting all of this into $y - y_1 = m(x - x_1)$,

$$\begin{aligned} a^{\frac{1}{3}} - 0 &= \tfrac{1}{3}a^{-\frac{2}{3}}\left(a + \tfrac{2}{5}\right) \\ \Longrightarrow a &= \tfrac{1}{3}\left(a + \tfrac{2}{5}\right) \\ \Longrightarrow a &= \tfrac{1}{5}. \end{aligned}$$

3999. The sum to be differentiated is a geometric series with first term a and common ratio r. Hence, its sum to infinity is

$$S_\infty = \frac{a}{1-r}$$

Considering both a and r as variables depending on t, we differentiate by the quotient rule:

$$\begin{aligned} &\frac{d}{dt}\left(\sum_{i=1}^{\infty} ar^{i-1}\right) \\ &= \frac{d}{dt}\left(\frac{a}{1-r}\right) \\ &\equiv \frac{\frac{da}{dt}(1-r) - a\left(-\frac{dr}{dt}\right)}{(1-r)^2} \\ &\equiv \frac{(1-r)\frac{da}{dt} + a\frac{dr}{dt}}{(1-r)^2}, \text{ as required.} \end{aligned}$$

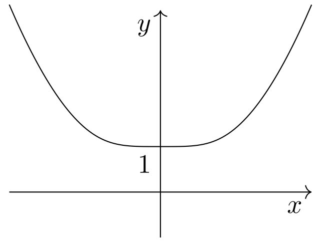
4000. (a) Differentiating,

$$\tfrac{dy}{dx} = \tfrac{1}{2}(x^4+1)^{-\frac{1}{2}} \cdot 4x^3.$$

Evaluating at $x = 0$, we have $\frac{dy}{dx} = 0$, so the curve is stationary. The second derivative is also zero; therefore, the second derivative test is inconclusive. However, $x^4 + 1$ is minimised at $x = 0$, and, since the square root function is increasing, $\sqrt{x^4+1}$ is minimised at $x = 0$.

(b) For large x, 1 is negligible compared to x^4. Hence, the graph approaches $y = \sqrt{x^4} \equiv x^2$.

(c) The graph is even, with the y axis as a line of symmetry, and is asymptotic to the parabola $y = x^2$ for large x.



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